Inverse Scattering for KPI equation

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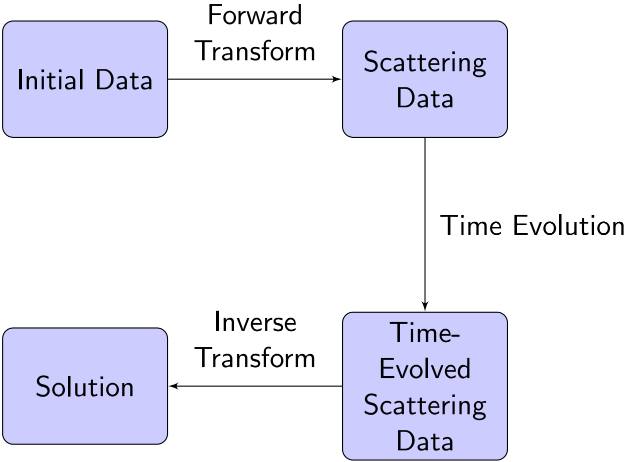
Schematic Description of Solving KPI by using IST



Lax Pair Representation for the KPI Equation

The KPI equation is given by

$$(u_t + 6uu_x + u_{xxx})_x = 3u_{yy} \tag{1}$$

Dryuma (1974) found a Lax pair for (1) in the following form:

$$i\psi_y + \psi_{xx} + u\psi = 0 \tag{2}$$

$$\psi_t + 4\psi_{xxx} + 6u\psi_x + 3\psi\left[u_x - i\int_{-\infty}^{x} u_y\, dx'\right] = 0 \tag{3}$$

where the KPI equation is the compatibility condition for (2) and (3).

Note: ψ is scaled by phase, i.e., $\psi \to \psi e^{i\lambda y}$ to eliminate the spectral parameter λ in the original Schrödinger equation.

A Lump Solution

There are two important aspects of the KPI equation compared to the KdV equation:

1. Soliton solutions of KdV are also solutions of KPI without depending on y but they are linearly unstable (Kadomtsev and Petviashvili, 1970).
2. The KPI equation also admits lump solutions that decay algebraically both in x and y (Bordag, Its, Manakov, Matveev and Zakharov, 1977, Ablowitz and Satsuma, 1978).

One lump solution is given by

$$u(x,y,t) = 2\partial_x^2 \ln\left[(x+X)^2 + Y^2\right] \tag{4}$$

where

$$X(y,t) = ay - 3(b^2 - a^2)t + x_0,$$
$$Y^2(y,t) = b^2(y + 6at + y_0)^2 + b^{-2}, \quad Y \geq 0$$

Solving KPI as an IVP

Zakharov and Manakov (1979) and Manakov (1981) developed an inverse scattering formalism to solve (1):

- They considered (2) as a scattering problem and obtained a linear integral equation of Gel'fand-Levitan-Marchenko type.
- However, the class of initial data was not specified, other than saying $u(x, y, t = 0)$ must vanish rapidly as $x^2 + y^2 \to \infty$.

Manakov, Santini and Takhtajan (1980) showed that lumps solutions do not evolve from initial data for which these methods are valid.

Question: Were lumps excluded by some limitation of Manakov's method, or are they unstable in some sense?

Relation of A Lump Solution to Initial Data

Denote the initial data of (1) by $u(x, y)$. Fourier transform of $u(x, y)$ in x variable is given by

$$\hat{u}(m, y) = \int u(x, y) e^{-imx} dx$$

so that

$$u(x, y) = \frac{1}{2\pi} \int \hat{u}(m, y) e^{imx} dm$$

Assume that

$$U(\infty) = \frac{1}{2\pi} \int \int |\hat{u}(m, y)| dm dy < \infty \tag{5}$$

Taking the Fourier transform of a lump solution in (4) at $t = 0$,

$$\hat{u}(m, y) = 4\pi |m| e^{-|m|Y + imX} \tag{6}$$

so that

$$U(\infty) = 4\pi \tag{7}$$

*Scattering solutions in the direct scattering problem are defined iteratively if

$$U(\infty) < 1 \tag{8}$$

Analogy with Modified KdV Equation

The modified KdV equation is given by

$$v_t + 6v^2 v_x + v_{xxx} = 0$$

Then every soliton satisfies

$$\int |v| dx = \pi$$

The Gel'fand-Levitan equation can be solved iteratively (Ablowitz, Kaup, Newell and Segur, 1974) if the initial data satisfies

$$\int |v| < 0.904$$

So, the solution evolving from this initial data does not contain solitons.

More Restrictions on Initial Data

First, assume that for each fixed t, u and its x derivatives vanish as $x \to -\infty$. Then, integrating (1) in x,

$$u_t + 6uu_x + u_{xxx} \sim 3\partial_y^2 \int_{-\infty}^{x} u dx'$$

Observe that u_t also vanishes as $x \to -\infty$. Then, assume that u and its x derivatives vanish as $x \to +\infty$ for each fixed t. So,

$$u_t \sim 3\partial_y^2 \int u dx'$$

as $x \to +\infty$. If we require u_t to vanish as $x \to +\infty$, we need

$$\int u(x, y, t)dx = A(t)y + B(t) \tag{9}$$

Finally, assume that

$$\overline{U} = \int \int |u(x, y)| dxdy < \infty \tag{10}$$

More Restrictions on Initial Data (ct'd)

At $t = 0$, integrating (9) in y over $[-R, R]$ for some $R > 0$,

$$\iint u(x, y)dxdy = 2B(0)R$$

Then using (10), we obtain $2|B(0)|R < \infty$ for any $R > 0$. So, $B(0) = 0$. Similarly, at $t = 0$, integrating (9) in y over $[0, R]$ gives $A(0) = 0$. Thus,

$$\int u(x, y)dx = 0 \tag{11}$$

The method discussed requires the restriction (11) on the initial data in addition to (5).

Left and Right Scattering Solutions

Since $u(x, y) \to 0$ as $y \to \pm\infty$, (2) has an asymptotic solution

$$\psi(x, y; k) \sim e^{ikx - ik^2 y} \quad \text{as } y \to \pm\infty$$

Let $\psi(x, y; k) = e^{ikx - ik^2 y}\mu(x, y; k)$, so that (2) becomes

$$i\mu_y + \mu_{xx} + 2ik\mu_x + u\mu = 0 \tag{12}$$

Let ψ_L and ψ_R be two solutions of (2) such that $\psi_L(x, y; k) \sim e^{ikx - ik^2 y}$ as $y \to -\infty$ and $\psi_R(x, y; k) \sim e^{ikx - ik^2 y}$ as y $\to +\infty$.

Now, let

$$\begin{aligned} \mu_L(x, y; k) &= 1 + \frac{i}{2\pi}\int_{-\infty}^{y}\int\int e^{i\phi}u(x', y')\mu_L(x', y'; k)\ dm\ dx'dy' \\ \mu_R(x, y; k) &= 1 - \frac{i}{2\pi}\int_{y}^{\infty}\int\int e^{i\phi}u(x', y')\mu_L(x', y'; k)\ dm\ dx'dy' \end{aligned} \tag{13}$$

where

$$\phi = m(x - x') - m(m + 2k)(y - y')$$

so that $\mu_L \to 1$ as $y \to -\infty$ and $\mu_R \to 1$ as $y \to +\infty$.

Left and Right Scattering Solutions (ct'd)

Write $\mu_L = 1 + T_u(\mu_L)$ where,

$$T_u(\mu_L)(x,y;k) = \int_\infty^y \int \left[\frac{i}{2\pi}\int e^{i\phi}u(x',y')dm\right]\mu_L(x',y';k)dx'dy'$$

For u with sufficiently small norm, the resolvent operator $[I - T_u]^{-1}$ exists and it has a convergent Neuman series. Thus, $\mu_L = (I - T_u)^{-1}1 = \sum_{n=0}^{\infty} T_u^n 1$. Let $\mu_{L,n} = T_u^n 1$, so that

$$\mu_L(x,y;k) = 1 + \sum_{n=1}^{\infty} \mu_{L,n}(x,y;k)$$

Substituting this into (13),

$$\mu_{L,1}(x,y;k) = \frac{i}{2\pi}\int_{-\infty}^{y}\int\int e^{i\phi}u(x',y')\ dm\ dx'\ dy'$$

and for any $n \geq 1$

$$\mu_{L,n+1}(x,y;k) = \frac{i}{2\pi}\int_{-\infty}^{y}\int\int e^{i\phi}u(x',y')\mu_{L,n}(x',y';k)\ dm\ dx'\ dy'$$

Left and Right Scattering Solutions (ct'd)

Assuming that $\mu_{L,n}(x,y;k)$ has a Fourier transform in x, $\mu_{L,n}(m,y;k)$,

$$\widehat{\mu}_{L,1}(x,y;k) = i\int_{-\infty}^{y} e^{-im(m+2k)(y-y')}\widehat{u}(m,y')dy'$$

and for any $n \geq 1$

$$\widehat{\mu}_{L,n+1}(x,y;k) = \frac{i}{2\pi}\int_{-\infty}^{y} e^{-im(m+2k)(y-y')}(u * \mu_{L,n})(m,y';k)dy' \qquad (14)$$

Let

$$U(y) = \frac{1}{2\pi}\int_{-\infty}^{y}\int |\widehat{u}(m,y)|\ dm\ dy'$$

Then, by (14),

$$\frac{1}{2\pi}\int |\widehat{\mu}_{L,1}(m,y;k)|dm \leq U(y) \qquad (15)$$

Left and Right Scattering Solutions (ct'd)

Hence, it follows from (14) and (15) that

$$\frac{1}{2\pi}\int |\widehat{\mu}_{L,n}(m,y;k)|dm \leq \frac{U(y)^n}{n!}$$

Note that $|\mu_{L,n}(x,y;k)| \leq \frac{1}{2\pi}\int |\widehat{\mu}_{L,n}(m,y,k)|dm$. Then,

$$|\mu_L(x,y;k)| \leq 1 + \sum_{n=1}^{\infty} \frac{U(y)^n}{n!} = e^{U(y)} < e^{U(\infty)}$$

Similarly,

$$|\mu_R(x,y;k)| < e^{U(\infty)}$$

Thus, if $U(\infty) < \infty$, then (13) has a unique solution for all $x, y, k \in \mathbb{R}$.

Scattering Kernel S

Define a scattering kernel as

$$S(k, k+m) = -\frac{i}{2\pi} \iint e^{-imx' + im(m+2k)y'} u(x', y') \mu_R(x', y'; k) dx' dy' \quad (16)$$

We wish to show that

$$\mu_R(x, y; k) = \mu_L(x, y; k) + \int S(k, k+m) \mu_L(x, y; k+m) e^{imx - im(m+2k)y} dm$$

or

$$\psi_R(x, y; k) = \psi_L(x, y; k) + \int S(k, l) \psi_L(x, y; l) dl \quad (17)$$

Rewrite (13) as

$$\mu_{R,L} = 1 + G_{R,L} * (u \mu_{R,L})$$

where the Green's functions are

$$G_{R,L}(x, y; k) = \mp \frac{i}{2\pi} \theta(\pm y) \int e^{imx - im(m+2k)y} dm$$

Scattering Kernel S (ct'd)

Note that

$$[G_R - G_L](x,y;k) = -\frac{i}{2\pi}\int e^{imx-im(m+2k)y}\,dm \tag{18}$$

Let $\Delta\mu = \mu_R - \mu_L$. So, $\Delta\mu = G_R * (u\mu_R) - G_L * (u\mu_L)$. Then, rewrite $\Delta\mu$ as

$$\Delta\mu = (G_R - G_L) * (u\mu_R) + G_L * (u(\Delta\mu)) \tag{19}$$

Substituting (18) into (19),

$$\Delta\mu(x,y;k) = \int S(k,k+m)e^{imx-im(m+2k)y}\,dm + [G_L * (u\Delta\mu)](x,y;k) \tag{20}$$

For $U(\infty) < 1$, the resolvent operator $[I - G_L * (u\cdot)]^{-1}$ exists. So, we can solve (20) for $\Delta\mu$,

$$\Delta\mu(x,y;k) = \int S(k,k+m)\{[I - G_L * (u\cdot)]^{-1}e^{imx-im(m+2k)y}\}\,dm \tag{21}$$

with

$$[I - G_L * (u\cdot)]^{-1} = \sum_{n=0}^{\infty}[G_L * (u\cdot)]^n$$

Scattering Kernel S (ct'd)

Comparing (21) to (17), it suffices to show that for any $n \in \mathbb{Z}^+$,

$$[G_L * (u\cdot)]^n e^{imx-im(m+2k)y} = \mu_{L,n}(x,y;k+m)e^{imx-im(m+2k)y} \tag{22}$$

so that

$$[I - G_*(u\cdot)]^{-1} e^{imx-im(m+2k)y]} = \mu_L(x,y;k+m)e^{imx-im(m+2k)y}$$

Note that the zero-th order term in (22) is $e^{imx-im(m+2k)y}$.

We calculate the first-order term to be

$$\left[G_L * \left(ue^{im\cdot-im(m+2k)\cdot}\right)\right](x,y;k) = e^{imx-im(m+2k)y}\mu_{L,1}(x,y;k+m)$$

By induction, we obtain (22), so that we prove the jump relation (17).

Comments on Left and Right Scattering Solutions

To solve the inverse scattering problem, we require that

1. the scattering kernel evolves linearly in time.
2. The scattering solutions involved are analytic in k in appropriate half-planes.

However, μ_L and μ_R generally are not analytic in k. The integrals in (13) are defined only for real k if $u(x, y)$ is real because $y - y'$ is unbounded and m takes both negative and positive values.

So, μ_L and μ_R are not appropriate scattering solutions for the inverse problem if the initial data is real.

Alternative Set of Scattering Solutions

Define at $t = 0$,

$$\mu^{\uparrow}(x,y;k) = 1 - \frac{i}{2\pi}\int_y^{\infty}\int_0^{\infty}\int e^{i\phi}u(x',y')\mu^{\uparrow}(x',y';k)\;dx'\;dm\;dy' + \frac{i}{2\pi}\int_{-\infty}^{y}\int_{-\infty}^{0}\int e^{i\phi}u(x',y')\mu^{\uparrow}(x',y';k)\;dx'\;dm\;dy'$$

$$\mu^{\downarrow}(x,y;k) = 1 - \frac{i}{2\pi}\int_y^{\infty}\int_{-\infty}^{0}\int e^{i\phi}u(x',y')\mu^{\downarrow}(x',y';k)\;dx'\;dm\;dy' + \frac{i}{2\pi}\int_{-\infty}^{y}\int_0^{\infty}\int e^{i\phi}u(x',y')\mu^{\downarrow}(x',y';k)\;dx'\;dm\;dy' \quad (23)$$

Note that $\mu^{\uparrow}$ can be extended to $\text{Im}(k) > 0$ and $\mu^{\downarrow}$ to $\text{Im}(k) < 0$.

Upper and Lower Scattering Solutions

Let us solve (23) iteratively similar to what we did before.
Let $\mu^{\uparrow} = 1 + \sum_{n=1}^{\infty} \mu_n^{\uparrow}$.
Then by (23),

$$\mu_1^{\uparrow}(x, y; k) = \frac{i}{2\pi} \int_{-\infty}^{y} \int_{-\infty}^{0} \int e^{i\phi} u(x', y')\ dx'\ dm\ dy' - \frac{i}{2\pi} \int_{y}^{\infty} \int_{0}^{\infty} \int e^{i\phi} u(x', y')\ dx'\ dm\ dy'$$

and for any $n \geq 1$,

$$\mu_{n+1}^{\uparrow}(x, y; k) = \frac{i}{2\pi} \int_{-\infty}^{y} \int_{-\infty}^{0} \int e^{i\phi} u(x', y') \mu_n^{\uparrow}(x', y'; k)\ dx'\ dm\ dy' - \frac{i}{2\pi} \int_{y}^{\infty} \int_{0}^{\infty} \int e^{i\phi} u(x', y') \mu_n^{\uparrow}(x', y'; k)\ dx'\ dm\ dy'$$

Upper and Lower Scattering Solutions (ct'd)

Taking the Fourier transform of $\mu_n^{\uparrow}(x, y; k)$ in x,

$$\widehat{\mu}_1^{\uparrow}(m, y; k) = i \int e^{-i(m+2k)(y-y')} \widehat{u}(m, y') \cdot \left[\theta(y - y')\theta(-m) - \theta(-(y - y'))\theta(m)\right] dy'$$

$$\widehat{\mu}_{n+1}^{\uparrow}(m, y; k) = \frac{i}{2\pi} \int e^{-i(m+2k)(y-y')} (m, y')(u * \mu_n^{\uparrow})(m, y'; k) \cdot \left[\theta(y - y')\theta(-m) - \theta(-(y - y'))\theta(m)\right] dy' \quad (24)$$

Then, by (24),

$$\frac{1}{2\pi} \int |\widehat{\mu}_1^{\uparrow}(m, y; k)| dm \le \frac{1}{2\pi} \left[\int_{-\infty}^{y} \int_{-\infty}^{0} |\widehat{u}(m, y')| \; dm \; dy' + \int_{y}^{\infty} \int_{0}^{\infty} |\widehat{u}(m, y')| \; dm \; dy' \right] \quad (25)$$

Note that since $u(x, y)$ is real, $\overline{\widehat{u}(-m, y)} = \widehat{u}(m, y)$.

Upper and Lower Scattering Solutions (ct'd)

Taking m to $-m$ in the first term on the right hand side of (25),

$$\frac{1}{2\pi}\int|\widehat{\mu}_1^{\uparrow}(m,y;k)|dm \leq \frac{1}{2\pi}\int\int_0^{\infty}|\widehat{u}(m,y')|\ dm\ dy' = \frac{1}{2}U(\infty) \tag{26}$$

Hence, it follows from (24) and (26) that

$$\frac{1}{2\pi}\int|\widehat{\mu}_n^{\uparrow}(m,y;k)|dm \leq \frac{1}{2}U(\infty)^n \tag{27}$$

Thus, if $U(\infty) < 1$, then (23) has a unique solution for all $x, y, k \in \mathbb{R}$, which is uniformly bounded by

$$|\mu^{\uparrow}(x,y;k)| \leq 1 + \frac{1}{2}\sum_{n=1}^{\infty}U(\infty)^n = \frac{2-U(\infty)}{2(1-U(\infty))}$$

Differentiating $\mu^{\uparrow}(x,y;k)$ in (23) with respect to k, we observe that the nonhomogeneous term is defined for $\mathrm{Im}(k) > 0$ because if $m(y-y') < 0$, then

$$|2im(y-y')e^{-2ikm(y-y')}| \leq \frac{1}{\mathrm{Im}(k)e}$$

So, $\mu^{\uparrow}(x,y;k)$ is analytic in $\mathrm{Im}(k) > 0$.

Scattering Solutions for One Lump

Consider one lump solution (4) at $t = 0$ so that $U(\infty) < 1$ does not hold. Ablowitz and Fokas showed that at $t = 0$ the scattering solutions are given by

$$\mu(x, y; k_\pm) = \frac{c_\pm}{x + Z} + \frac{d_\pm}{x + Z^*}$$

which satisfies (13) and vanishes as $x^2 + y^2 \to \infty$, where

$$Z = X(y) + iY(y), \quad c_\pm = 1 \mp \frac{b^2(y + y_0) - i}{bY}, \quad d_\pm = c_\mp, \quad 2k_\pm = -a \pm ib$$

One can show that $\mu(x, y; k_\pm)$ are homogeneous solutions of (23), so (23) cannot be solved iteratively if the initial data is one lump, i.e., (4) at $t = 0$.

Asymptotic Behavior of Upper Scattering Solution

Integrating by parts in y', one can show that as $\mathrm{Im}(k) \to \infty$,

$$\mu^{\uparrow}(x, y; k) \sim 1 + \frac{1}{4\pi k} \int \frac{\widehat{u}(m, y)}{m} dm + o(|k|^{-1}) \tag{28}$$

Note that (28) is valid if $\widehat{u}(0, y) = 0$, which is the same as (11).
Also, note that if we can recover $\mu^{\uparrow}$ from the scattering data in the inverse problem, then u can be obtained from $\mu^{\uparrow}$ using (28).

Asymptotic Behavior of Upper Scattering Solution (ct'd)

Alternatively, (28) can be obtained by (12).
Let $\mu = 1 + \nu$, and rewrite (12) as

$$i\nu_y + \nu_{xx} + 2ik\nu_x + u\nu + u = 0$$

Note that as $\text{Im}(k) \to \infty$, either $\nu \to 0$ or $\nu = 0$. The second gives trivial solution $u = 0$.
If ν and its derivatives vanish as $\text{Im}(k) \to 0$, then we obtain

$$2ik\nu_x + u \sim 0 \tag{29}$$

Taking the Fourier transform of (29), then solving for v gives the leading term in (28).

Jump Relation Between Upper and Lower Scattering Solutions

We wish to show that

$$\mu^{\uparrow}(x,y;k)-\mu^{\downarrow}(x,y;k)=\int F(k,l)\mu^{\downarrow}(x,y;l)e^{i(l-k)x-i(l^2-k^2)y}dl$$

or

$$\psi^{\uparrow}(x,y;k)=\psi^{\downarrow}(x,y;k)+\int F(k,l)\psi^{\downarrow}(x,y;l)dl \qquad (30)$$

Assume that $F(k,\cdot)\in L^1(\mathbb{R})$.
Rewrite (23) as

$$\mu^{\uparrow,\downarrow}=1+G^{\uparrow,\downarrow}*(u\mu^{\uparrow,\downarrow})$$

where the Green's functions are

$$G^{\uparrow,\downarrow}(x,y;k)=\frac{i}{2\pi}\int\left[\theta(y)\theta(\mp m)-\theta(-y)\theta(\pm)\right]e^{imx-im(m+2k)y}dm$$

Jump Relation Between Upper and Lower Scattering Solutions (ct'd)

Note that

$$[G^{\uparrow} - G^{\downarrow}](x, y; k) = \frac{i}{2\pi} \int sgn(m) e^{imx - im(m+2k)y} dm \tag{31}$$

Let $\Delta\mu = \mu^{\uparrow} - \mu^{\downarrow}$. So, $\Delta\mu = G^{\uparrow} * (u\mu^{\uparrow}) - G^{\downarrow} * (u\mu^{\downarrow})$. Then, rewrite $\Delta\mu$ as

$$\Delta\mu = (G^{\uparrow} - G^{\downarrow}) * (u\mu^{\uparrow}) + G^{\downarrow} * (u(\Delta\mu)) \tag{32}$$

Substituting (31) into (32) and (30) into both sides of (32), we obtain

$$\int F(k, l)\mu^{\downarrow}(x, y; l) e^{i[(l-k)x - (l^2 - k^2)y]} dl$$
$$= \int T(k, l) e^{i(l-k)x - i(l^2-k^2)y} dl +$$
$$\iint G^{\downarrow}(x - x', y - y'; k) u(x', y') \int F(k, l)\mu^{\downarrow}(x', y'; l) e^{i(l-k)x' - i(l^2-k^2)y'} dl dx' dy' \tag{33}$$

Jump Relation Between Upper and Lower Scattering Solutions (ct'd)

where

$$T(k,k+m) = -\frac{i}{2\pi}\,\text{sgn}(m)\iint e^{-imx'+im(m+2k)y'}u(x',y')\mu^{\uparrow}(x',y';k)dx'dy' \tag{34}$$

Rewriting (23) for $\mu^{\downarrow}(x,y;l)$, multiplying by $F(k,l)e^{i[(l-k)x-(l^2-k^2)y]}$ and itegrating in l,

$$\begin{aligned}&\int F(k,l)\mu^{\downarrow}(x,y;l)e^{i(l-k)x-i(l^2-k^2)y}dl\\ &=\int F(k,l)e^{i(l-k)x-i(l^2-k^2)y}dl\\ &+\int\int\int G^{\uparrow}(x-x',y-y';l)u(x',y')\mu^{\downarrow}(x',y';l)dx'dy'F(k,l)e^{i(l-k)x-i(l^2-k^2)y}dl\end{aligned} \tag{35}$$

Jump Relation Between Upper and Lower Scattering Solutions (ct'd)

Subtracting (35) from (33) and taking the Fourier transform,

$$F(k,l) - T(k,l) + \int_{-\infty}^{l} T_1(p,l)F(k,p)dp = 0, \quad \text{if } k > l$$

$$F(k,l) - T(k,l) - \int_{l}^{\infty} T_1(p,l)F(k,p)dp = 0, \quad \text{if } k < l \qquad (36)$$

where

$$T_1(k,k+m) = \frac{i}{2\pi}\,\text{sgn}(m) \iint e^{-imx'+im(m+2k)y'} u(x',y')\mu^{\downarrow}(x',y';k)dx'dy' \qquad (37)$$

Jump Relation Between Upper and Lower Scattering Solutions (ct'd)

Let $l = k + m$ in (34) and rewrite as

$$
\begin{aligned}
T(k,l) = &-\frac{i}{2\pi}\operatorname{sgn}(l-k)\Bigg[\iint e^{-i(l-k)x'+i(l^2-k^2)y'}u(x',y')dx'dy' \\
&+\iint e^{-i(l-k)x'+i(l^2-k^2)y'}u(x',y')\sum_{n=1}^{\infty}\mu_n^{\uparrow}(x',y')dx'dy'\Bigg] \\
&= -\frac{i}{2\pi}\operatorname{sgn}(l-k)\Bigg[\int e^{i(l^2-k^2)y'}\widehat{u}(l-k,y')dy' \\
&+\frac{1}{2\pi}\iint e^{i(l^2-k^2)y'}\widehat{u}(m',y')\sum_{n=1}^{\infty}\widehat{\mu}^{\uparrow}(l-k-m',y')dy'dm'\Bigg]
\end{aligned}
$$

Jump Relation Between Upper and Lower Scattering Solutions (ct'd)

Now, note that since we assumed $U(\infty) < 1$, then

$$\begin{aligned}\|T(k,\cdot)\|_{L^1(\mathbb{R})} &\leq \frac{1}{2\pi}\iint |\widehat{u}(l-k,y')|dy'dl \\ &+ \frac{1}{2\pi}\iint |\widehat{u}(m',y')|\left[\sum_{n=1}^{\infty}\frac{1}{2\pi}\int |\widehat{\mu}_n^{\uparrow}(l-k-m')|dl\right]dm'dy' \\ &\leq U(\infty)\left[1+\frac{1}{2}\sum_{n=1}^{\infty}U(\infty)^n\right] \quad \text{by (5) and (27)} \\ &\leq \frac{U(\infty)(2-U(\infty))}{2(1-U(\infty))} < \infty\end{aligned}$$

Similarly, $\|T_1(k,\cdot)\|_{L^1(\mathbb{R})} \leq \dfrac{U(\infty)(2-U(\infty))}{2(1-U(\infty))} < \infty$.

Jump Relation Between Upper and Lower Scattering Solutions (ct'd)

However, we show that we need $\|T_1(k,\cdot)\|_{L^1(\mathbb{R})} < 1$, i.e., $U(\infty) < 2 - \sqrt{2}$, to have $F(k,\cdot) \in L^1(\mathbb{R})$, so that $F(k,l)$ is defined by (36) for each fixed k.

Assume that $\|T_1(k,\cdot)\|_{L^1(\mathbb{R})} < 1$ and we have that $T(k,\cdot) \in L^1(\mathbb{R})$. We wish to show that $F(k,\cdot) \in L^1(\mathbb{R})$.

For $k > l$, rewrite (36) as

$$F(k,l) - T(k,l) + S(F)(k,l) = 0$$

where

$$S(F)(k,l) = \int_{-\infty}^{l} T_1(p,l)F(k,p)dp$$

Jump Relation Between Upper and Lower Scattering Solutions (ct'd)

It suffices to show that

$$\|S(F)(k,\cdot)\|_{L^1(-\infty,k)} < \|F(k,\cdot)\|_{L^1(-\infty,k)}$$

Note that

$$\begin{aligned}\|S(F)(k,\cdot)\|_{L^1(-\infty,k)} &\leq \int_{-\infty}^{k}\int_{-\infty}^{l}|T_1(p,l)||F(k,p)|\ dp\ dl\\ &\leq \int_{-\infty}^{k}|F(k,p)|\int_{p}^{k}|T_1(p,l)|\ dl\ dp\\ &\leq \|T_1(p.\cdot)\|_{L^1(-\infty,k)}\|F\|_{L^1(-\infty,k)}\\ &\leq \|F\|_{L^1(-\infty,k)}\end{aligned}$$

This shows that $F(k,\cdot) \in L^1(-\infty,k)$.
Similarly, for $k < l$, we obtain $F(k,\cdot) \in L^1(k,\infty)$, so that $F(k,\cdot) \in L^1(\mathbb{R})$.

Jump Relation Between Upper and Lower Scattering Solutions (ct'd)

Finally, if (10), $u \in L^1(\mathbb{R}^2)$ with $\|u\|_{L^1(\mathbb{R}^2)} = \overline{U}$ also holds, then

$$|T(k,l)| \leq \frac{1}{2\pi} \frac{2 - U(\infty)}{2(1 - U(\infty))} \overline{U} \quad \text{and} \quad |T_1(k,l)| \leq \frac{1}{2\pi} \frac{2 - U(\infty)}{2(1 - U(\infty))} \overline{U}$$

Thus, $F(k,l)$ is defined pointwise by (36).

Therefore, given $F(k,l)$, defined by (36), $\psi^{\uparrow}$ and $\psi^{\downarrow}$ are related by (30).

Hence, the direct scattering problem at $t = 0$ is complete.

Relation between Scattering Kernels S and T

Note that if $u(x, y)$ is real and ψ_1 and ψ_2 are any two solutions of (2), then

$$i\partial_y[\psi_1\overline{\psi}_2] + \partial_x[(\psi_1)_x\overline{\psi}_2 - \psi_1(\overline{\psi}_2)_x] = 0 \tag{38}$$

Observe that $\int \psi_1\overline{\psi}_2 dx$ is y independent if the boundary terms vanish after integrating (38) first in y, then in x.
However, the boundary terms do not vanish for any of ψ_R,ψ_L, $\psi^\uparrow$, $\psi^\downarrow$.
Note that as $y \to \infty$, $\psi_R(x, y; l) \sim e^{ilx-il^2y}$ and by (23)

$$\psi^\uparrow(x, y; k) \sim e^{ikx-iky^2}\left[1 + \frac{i}{2\pi}\int\int_{-\infty}^{0}\int e^{i\phi}u(x', y')\mu^\uparrow(x', y'; k)dx'\ dm\ dy'\right]$$

Then, using the Dominated Convergence Theorem, as $y \to \infty$,

$$\int\left[\psi^\uparrow(x, y; k)\overline{\psi_R(x, y; k)} - e^{i(k-l)x-i(k^2-l^2)y}\right]dx$$
$$\to 2\pi\theta(k-l)T(k, l) \tag{39}$$

Relation between Scattering Kernels S and T (ct'd)

Similarly, as $y \to -\infty$,

$$\int \left[\psi^{\uparrow}(x,y;k)\overline{\psi_L(x,y;k)} - e^{i(k-l)x - i(k^2-l^2)y} \right] dx \to 2\pi\theta(l-k)T(k,l) \quad (40)$$

Segur asserts that if we compute $\int \psi^{\uparrow}(x,y;k)\left[\overline{\psi_R(x,y;l)} - \overline{\psi_L(x,y;l)}\right] dx$ using (17), we can obtain the desired relation between T and S,

$$T(k,l)\,\mathrm{sgn}(k-l) = \overline{S}(l,k) + \int_0^{\infty} \overline{S}(l,k+m)T(k,k+m) \quad (41)$$

Time Evolution of Scattering Kernel S

The time evolution of $\psi(x, y, t; k)$ is given by

$$M_k\psi = [\partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x - 3i(\partial_x^{-1}u_y) + \alpha(k)]\psi = 0$$

Recall that as $y \to \pm\infty$, $\psi(x, y; k) \sim e^{ikx - ik^2 y}$, and u with its derivatives vanish. So,

$$M_k\psi \sim [\partial_t + 4\partial_x^3 + \alpha(k)]e^{ikx - ik^2 y} \quad \text{as } y \to \pm\infty$$

gives $\alpha(k) = 4ik^3$.

Consider the time-dependent version of (17)

$$\psi_R(x, y, t; k) = \psi_L(x, y, t; k) + \int S(k, l, t)\psi_L(x, y, t; l)dl \qquad (42)$$

Time Evolution of Scattering Kernel S (ct'd)

Note that $M_k\psi_R = M_k\psi_L = 0$. So, applying M_k to both sides of (42) and taking the limit as $y \to -\infty$, we obtain

$$\begin{aligned} 0 &= \int M_k[S(k,l,t)\psi_L(x,y,t;l)]dl \\ &\sim \int [\partial_t + 4\partial_x^3 + \alpha(k)][S(k,l,t)e^{ilx-il^2y}]dl \\ &= \int \left[[(\partial_t + \alpha(k))S(k,l,t)]e^{ilx-il^2y} + S(k,l,t)[(\partial_t + 4\partial_x^3)e^{ilx-il^2y}]\right] dl \\ &= \int \left[\partial_t S(k,l,t) + 4i(k^3 - l^3)S(k,l,t)\right] e^{ilx-il^2y} dl \qquad (43) \end{aligned}$$

Let $g(k,l,t) = [\partial_t S(k,l,t) + 4i(k^3 - l^3)S(k,l,t)]e^{-il^2y}$. Multiplying both sides of (43) by a test function $\varphi \in \mathcal{S}(\mathbb{R})$ and integrating in x,

$$0 = \int g(k,l,t)\widehat{\varphi}(l)dl \quad \text{for any } \varphi \in \mathcal{S}(\mathbb{R})$$

So, $g(k,l,t) = 0$, i.e., $\partial_t S(k,l,t) + 4i(k^3 - l^3)S(k,l,t) = 0$. Thus,

$$S(k,l,t) = S(k,l)e^{4i(l^3-k^3)t} \qquad (44)$$

Time Evolution of Scattering Kernels T, T_1 and F (ct'd)

Since $S^*(k,l,t)$ evolves in accordance with (44), then

$$T(k,l,t) = T(k,l)e^{4i(l^3-k^3)t} \tag{45}$$

satisfies the time-dependent version of (41).
Similarly,

$$T_1(k,l,t) = T_1(k,l)e^{4i(l^3-k^3)t} \tag{46}$$

Then,

$$F(k,l,t) = F(k,l)e^{4i(l^3-k^3)t} \tag{47}$$

satisfies the time-dependent version of (36).

Thus, all of the scattering kernels S, T, T_1, F evolve linearly in time.

Recovering a Solution of KPI via Inverse Scattering

Comments on Solving for $u(x, y, t)$ via Inverse Scattering:

- $F(k, l, t)$ is given in terms of the initial data $u(x, y)$ via (36) and (47).
- $u(x, y, t)$ can be recovered from $\mu^{\uparrow}(x, y, t)$ via (28) or (12).
- The main problem is to recover $\mu^{\uparrow}(x, y, t)$ in terms of $F(k, l, t)$ via (30).

A formal procedure for solving (30) (Manakov, 1981) assumes triangular representation given by

$$\psi^{\downarrow}(x, y; k) = e^{ikx - ik^2 y} + \int_{-\infty}^{x} K(x, z, y) e^{ikz - ik^2 y} dz \tag{48}$$

If (48) exists, then (30) can be reduced to a linear integral equation of Gel'fand-Levitan-Marchenko type.

Finally, if such K exists, substituting (48) into (2), we obtain

$$u(x, y) = -2 \frac{\partial K(x, x, y)}{\partial x}$$

so that we do not need (28) to recover the solution.

Questions on Justification of Manakov's Procedure

- Does triangular representation in (48) exit?
  Gel'fand and Levitan (1951) showed explicitely that their kernel corresponding to K in (48) exits using the theory of hyperbolic pdes but no such proof provided by Manakov.
- Are further restrictions on the initial data required to assure a unique solution of the Gel'fand-Levitan type equation?

Author deferred further analysis of the inverse problem to a later paper, which presumably was not published, in which there is no need for the initial data to be small, so that lump solutions are not excluded a priori.

References

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